

On the q -Laplace Transform

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Abstract

We introduce here the q -Laplace transform as a new weapon in Tsallis' arsenal, discussing its main properties and analyzing some examples. The q -Gaussian instance receives special consideration. Also, we derive the q -partition function from the q -Laplace transform.

KEYWORDS: q -Laplace transform, tempered ultradistributions, complex-plane generalization, one-to-one character.

1 Introduction

1.1 The Laplace transform

The Laplace transform, introduced by Laplace in his work on probability theory, is a widely used integral transform with many applications in physics and engineering. It is a linear operator acting on a function $f(t)$ with a real argument t that transforms it to a function $F(s)$ with complex argument s ,

$$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} dt e^{-st} f(t); \text{ } s \text{ complex.} \quad (1.1)$$

This transformation is essentially bijective for the majority of practical uses; the respective pairs $f(t)$, $L(s)$ are matched in tables. The Laplace transform has the useful property that many relationships and operations over the originals $f(t)$ correspond to simpler relationships and operations over the images $L(s)$. The Laplace transform is related to the Fourier transform (FT), but whereas the FT transform expresses a function or signal as a series of modes of vibration (frequencies), the Laplace one resolves a function into its moments. Customarily, in speaking of the Laplace transform without qualification one means the unilateral or one-sided transform. The Laplace transform can be alternatively defined as the bilateral, or two-sided one,

by extending the limits of integration to be the entire real axis. If that is done the common unilateral transform simply becomes a special case of the bilateral transform where the definition of the function being transformed is multiplied by the Heaviside step function.

1.2 q-Statistical theory

Nonextensive statistical mechanics (NEXT) [3, 4, 5], an extension of the standard Boltzmann-Gibbs (BG) one, is used in variegated scientific areas. NEXT is based on a nonadditive (although extensive [6]) information measure characterized by the real index q (with $q = 1$ recovering the BG entropy). It has been employed in diverse scenarios such as cold atoms in dissipative optical lattices [7], dusty plasmas [8], trapped ions [9], spin-glasses [10], turbulence [11], self-organized criticality [12], high-energy experiments at LHC/CMS/CERN [13] and RHIC/PHENIX/Brookhaven [14], low-dimensional dissipative maps [15], finance [16], galaxies [17], Fokker-Planck equation's applications [18], etc. A typical NEXT feature is that it can be advantageously cast using appropriate q -generalizations of standard mathematical concepts [19]. Included are, for instance, the logarithm and exponential functions, addition and multiplication, etc.

1.3 Our aims

Here we add, by recourse to ultradistributions (see Appendix) the Laplace transform tool to such armory. It is to be pointed out that, quite recently, an alternative form of the q -Laplace transform (q LP) has been advanced by Won Sang Chung [1], who uses for that purpose so-called q -sums, q -differences, q -products, and q -ratios, which renders the treatment rather abstract.

Our q LP is, instead, based on the ordinary version of the four elementary arithmetic operations. More importantly, Wong Sang Chung's definition does not follow the tenets of the pioneer paper by Umarov-Tsallis-Steinberg [2] that introduced the q Fourier transform. In particular, the function to be Wong-transformed does not contain a q -exponential argument, a crucial NEXT-aspect that must be respected so as to maintain the theory's consistency.

The reader is advised to peruse the Appendix before embarking into our discussion below.

2 Laplace transform from Fourier one

Before dealing with the q -Laplace transform it is convenient to show that the ordinary bilateral Laplace transform can be obtained from the complex Fourier transform. Thus, let (see Appendix for details and references)

- Λ_∞ be the space of distributions of exponential type,
- \mathcal{U} or $\mathcal{U}_{\mathbf{I}}$ the space of tempered ultradistributions,
- \mathcal{F} the Fourier transform connecting them, and
- H the Heaviside step function.

We have

$$\mathcal{F} : \Lambda_\infty \longrightarrow \mathcal{U}, \quad (2.2)$$

reading $\Im(k)$ is the imaginary part of k and $\Re(k)$ its real one)

$$\mathcal{F}(k) = H[\Im(k)] \int_0^\infty f(x) e^{ikx} dx - H[-\Im(k)] \int_{-\infty}^0 f(x) e^{ikx} dx. \quad (2.3)$$

The associated inversion formula is

$$f(x) = \frac{1}{2\pi} \oint_{\Gamma_F} \mathcal{F}(k) e^{-ikx} dk, \quad (2.4)$$

where the contour Γ_F surrounds all singularities of $\mathcal{F}(k)$ and runs parallel to the real axis from $-\infty$ to ∞ above it, and from ∞ to $-\infty$ below. Then,

making the change of variables $ik = -p$, the bilateral Laplace transform is obtained (see [20])

$$\mathcal{L} : \Lambda_\infty \longrightarrow \mathcal{U}_I \quad (2.5)$$

and given by

$$\mathcal{L}(p) = H[\Re(p)] \int_0^\infty f(x) e^{-px} dx - H[-\Re(p)] \int_{-\infty}^0 f(x) e^{-px} dx. \quad (2.6)$$

Let us insist: \mathcal{U}_I is the space of tempered ultradistributions and we have made the change of variables $p = -ik$. Now, the corresponding inversion formula is:

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma_L} \mathcal{L}(p) e^{px} dp, \quad (2.7)$$

where the contour Γ_L surrounds all singularities of $\mathcal{L}(p)$ and runs parallel to the imaginary axis from $-i\infty$ to $i\infty$ to the right of it, and from $i\infty$ to $-i\infty$ to the left. If we consider distributions of exponential type $f(x)$ such that $f(x) = 0$ for $x < 0$, we obtain the single Laplace transform

$$\mathcal{L}(p) = H[\Re(p)] \int_0^\infty f(x) e^{-px} dx, \quad (2.8)$$

whose inversion formula is

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma_{L+}^+} \mathcal{L}(p) e^{px} dp, \quad (2.9)$$

where Γ_L^+ is the right hand side of the path Γ_L .

3 q-Laplace transform

Let Ω stand the space of functions of the real variable x that are parameterized by a real parameters q . We have defined in ([21] - [23]) the q-Fourier transform

$$F : \Omega \longrightarrow \mathcal{U} \quad (3.10)$$

as

$$F(f)(k, q) = F(k, q) = [H(q - 1) - H(q - 2)] \times \left\{ H[\Im(k)] \int_0^\infty f(x) \{1 + i(1 - q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}, dx - H[-\Im(k)] \int_{-\infty}^0 f(x) \{1 + i(1 - q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} dx \right\}. \quad (3.11)$$

and its inverse transform as

$$f(x) = \frac{1}{2\pi} \oint_{\Gamma_F} \left[\lim_{\epsilon \rightarrow 0^+} \int_1^2 F(k, q) \delta(q - 1 - \epsilon) dq \right] e^{-ikx} dk. \quad (3.12)$$

Let Ω_I be the space of functions of the real variable x

$$\Omega_I = \{f(x)/f(x) \in \Omega_I^+ \cap \Omega_I^-\}, \quad (3.13)$$

where

$$\Omega_I^+ = \left\{ f(x)/f(x) \{1 - (1 - q)px[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^+]; \right.$$

$$f(x) \geq 0; |f(x)| \leq |x|^s g(x) e^{ax}; s, a \in \mathbb{R}^+; p \in \mathbb{Z}; \Re(p) \geq 0$$

$$1 \leq q < 2\}, \quad (3.14)$$

and

$$\Omega_I^- = \left\{ f(x)/f(x) \{1 - (1 - q)px[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^-]; \right.$$

$$f(x) \geq 0; |f(x)| \leq |x|^s g(x) e^{ax}; s, a \in \mathbb{R}^+; p \in \mathbb{Z}; \Re(p) \leq 0$$

$$1 \leq q < 2\}, \quad (3.15)$$

Making again the change $ik = -p$ we immediately obtain the bilateral q-Laplace transform L

$$L : \Omega_I \longrightarrow \mathcal{U}_I, \quad (3.16)$$

as

$$L(f)(p, q) = L(p, q) = [H(q - 1) - H(q - 2)] \times$$

$$\left\{ H[\Re(p)] \int_0^\infty f(x) \{1 - (1 - q)px[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}, dx - \right.$$

$$\left. H[-\Re(p)] \int_{-\infty}^0 f(x) \{1 - (1 - q)px[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} dx \right\}. \quad (3.17)$$

The corresponding inversion formula is easily found from (3.12)

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma_L} \left[\lim_{\epsilon \rightarrow 0^+} \int_1^2 L(p, q) \delta(q - 1 - \epsilon) dq \right] e^{px} dk. \quad (3.18)$$

If we consider $f \in \Omega_I$ such that $f(x) = 0$ for $x < 0$ we obtain the unilateral q-Laplace transform

$$L(p, q) = [H(q-1) - H(q-2)] \times \\ H[\Re(p)] \int_0^\infty f(x) \{1 - (1-q)px[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} dx, \quad (3.19)$$

and its inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma_{L+}} \left[\lim_{\epsilon \rightarrow 0^+} \int_1^2 L(p, q) \delta(q-1-\epsilon) dq \right] e^{px} dk. \quad (3.20)$$

We consider now functions $f_{q'} \in \Omega_I$ depending on the parameter q' with $1 \leq q' < 2$. We can define the singular q-Laplace transform

$$L_R : \Omega_I \longrightarrow \mathcal{U}_I, \quad (3.21)$$

as

$$L_R(f_{q'})(p, q') = L_R(p, q') = \lim_{q \rightarrow q'} L(f_{q'})(p, q) = L(f_{q'})(p, q) |_{q=q'}, \quad (3.22)$$

As is the case of the F_T q-Fourier transform (see [21], [23]), L_R is NOT one to one. To deal with such an issue we consider the set $\Lambda_{If_{q'}}$ given by

$$\Lambda_{If_{q'}} = \{g_{q'} \in \Omega_I / L_R(g_{q'})(k) = L_R(f_{q'})(k)\}, \quad (3.23)$$

and

$$\Lambda_I = \left\{ \Lambda_{If_{q'}} / f_{q'} \in \Omega_I \right\}. \quad (3.24)$$

Introducing the equivalence relation

$$g_{q'}(x) \sim f_{q'}(x) \iff g_{q'} \in \Lambda_{If_{q'}}, \quad (3.25)$$

and *the q-Laplace transform between equivalence classes*

$$L_{PR} : \Lambda_I \longrightarrow \mathcal{U}_I, \quad (3.26)$$

as

$$L_{PR}(\Lambda_{If_{q'}})(p, q') = L_{PR}(p, q') = L_R(f_{q'})(p, q'), \quad (3.27)$$

one finds that L_{PR} is one to one between equivalence classes and is the analog of F_{UTS} (the Umarov-Tsallis-Steinberg q-Fourier transform [2]) for the one to one q-Fourier transform.

4 Examples

We illustrate here with some examples the preceding developments. As a first one we consider the q-Laplace transform of the Heaviside's step function $f(x) = H(x)$. We have

$$L(p, q) = [H(q-1) - H(q-2)]H[\Re(p)] \int_0^\infty [1 + (q-1)px]^{\frac{1}{1-q}} dx. \quad (4.1)$$

Suitably manipulating (4.1) leads to

$$L(p, q) = [H(q-1) - H(q-2)] \frac{H[\Re(p)]}{(2-q)p}. \quad (4.2)$$

For $f(x) = H(-x)$ we have

$$L(p, q) = [H(q-1) - H(q-2)]H[-\Re(p)] \int_{-\infty}^0 [1 + (q-1)px]^{\frac{1}{1-q}} dx, \quad (4.3)$$

and, as a result,

$$L(p, q) = [H(q-1) - H(q-2)] \frac{H[-\Re(p)]}{(2-q)p}. \quad (4.4)$$

Taking now into account that $H(x) + H(-x) = 1$ we get for the q-Laplace transform of $f(x) = 1$

$$L(p, q) = [H(q-1) - H(q-2)] \frac{1}{(2-q)p}. \quad (4.5)$$

We evaluate now the transform of $f(x) = q' H(x)$, i.e.,

$$\begin{aligned} L(p, q, q') &= [H(q-1) - H(q-2)]H[\Re(p)] \times \\ &\int_0^{\infty} q' [1 + (q-1)pxq'^{(q-1)}]^{\frac{1}{1-q}} dx. \end{aligned} \quad (4.6)$$

One finds

$$L(p, q, q') = [H(q-1) - H(q-2)] \frac{q'^{(2-q)}}{2-q} \frac{H[\Re(p)]}{p}. \quad (4.7)$$

If we consider now the Laplace transform of the previous function we face

$$L_{PR}(p, q') = [H(q'-1) - H(q'-2)] \frac{q'^{(2-q')}}{2-q'} \frac{H[\Re(p)]}{p}. \quad (4.8)$$

As a last example we evaluate the transform of the function

$$f(x) = \begin{cases} \left(\frac{\lambda}{x}\right)^\beta ; & x \in [a, b] ; & 0 < a < b ; & \lambda > 0 \\ 0 ; & x \text{ outside } [a, b]. \end{cases} \quad (4.9)$$

One has

$$L(p, q) = [H(q-1) - H(q-2)]H[\Re(p)] \times \lambda^\beta \int_a^b x^{-\beta} \{1 - (1-q)p\lambda^{\beta(q-1)}x^{1-\beta(q-1)}\}^{\frac{1}{1-q}} dx \quad (4.10)$$

Following the steps of a similar calculation made in [21] allow us to obtain

$$\begin{aligned} L(p, q) = & [H(q-1) - H(q-2)]H[\Re(p)] \times \\ & \left\{ \left\{ H(q-1) - H\left[q - \left(1 + \frac{1}{\beta}\right)\right] \right\} \times \right. \\ & \quad \left. \frac{(q-1)\lambda^\beta}{(2-q)[(q-1)p\lambda^\beta]^{\frac{1}{q-1}}} \times \right. \\ & \quad \left. \left\{ a^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right. \right. \\ & \quad \left. \left. \frac{1}{(1-q)p\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right) - \right. \\ & \quad \left. b^{\frac{q-2}{q-1}} F\left(\frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1} + \frac{\beta(2-q)}{1-\beta(q-1)}; \right. \right. \\ & \quad \left. \left. \frac{1}{(1-q)p\lambda^{\beta(q-1)}b^{1-\beta(q-1)}}\right) \right\} + \\ & \quad \left\{ H\left[q - \left(1 + \frac{1}{\beta}\right)\right] - H(q-2) \right\} \frac{\lambda^\beta}{\beta-1} \times \\ & \quad \left\{ a^{1-\beta} F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1}; \right. \right. \end{aligned}$$

$$\begin{aligned}
& (1-q)p\lambda^{\beta(q-1)}a^{1-\beta(q-1)} - \\
& b^{1-\beta}F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1}; \right. \\
& \left. (1-q)p\lambda^{\beta(q-1)}b^{1-\beta(q-1)}\right)\}\}. \tag{4.11}
\end{aligned}$$

and, if we take $\beta = 1/(q-1)$, we get for our transform

$$L_{PR}(p, q) = H[\Re(p)] [H(q-1) - H(q-2)] [1 - (1-q)p\lambda]^{\frac{1}{1-q}}. \tag{4.12}$$

5 Series expansion of the q-Laplace transform

Consider the function

$$\{1 - (1-q)px[f(x)]^{q-1}\}^{\frac{1}{1-q}}.$$

Using the series expansions of the logarithm and the exponential function,

we can write

$$\begin{aligned}
\{1 - (1-q)px[f(x)]^{q-1}\}^{\frac{1}{1-q}} &= e^{\frac{1}{1-q} \ln\{1 - (1-q)px[f(x)]^{q-1}\}} = \\
&= e^{\frac{1}{1-q} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (q-1)^n (px)^n [f(x)]^{n(q-1)}} = \\
&= e^{\frac{1}{1-q} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (q-1)^n (px)^n e^{n(q-1) \ln f(x)}} = \\
&= e^{\frac{1}{1-q} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (q-1)^n (px)^n \sum_{m=0}^{\infty} \frac{n^m}{m!} (q-1)^m [\ln f(x)]^m} = \\
&= e^{\left[\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+1} n^{m-1}}{m!} (px)^n \ln^m[f(x)] (q-1)^{n+m-1} \right]} =
\end{aligned}$$

$$e^{\left\{ \sum_{n=0}^{\infty} \left[\sum_{m=0}^n \frac{(-1)^{n+1-m} (n+1-m)^{m-1}}{m!} (px)^{n-m+1} \ln^m[f(x)] \right] (q-1)^n \right\}}. \quad (5.1)$$

Let $g(x, p, n)$ be given by

$$g(x, k, n) = \sum_{m=0}^n \frac{(-1)^{n+1-m} (n-m+1)^{m-1}}{m!} (px)^{n-m+1} \ln^m[f(x)]. \quad (5.2)$$

Then,

$$\{1 - (1-q)px[f(x)]^{q-1}\} = h(x, p, q) = e^{\sum_{n=0}^{\infty} g(x, p, n)(q-1)^n}, \quad (5.3)$$

or

$$h(x, p, q) = e^{-px} e^{\sum_{n=1}^{\infty} g(x, p, n)(q-1)^n}. \quad (5.4)$$

Minding the expansion of the exponential function we have

$$e^{\sum_{n=1}^{\infty} g(x, p, n)(q-1)^n} = \sum_{l=0}^{\infty} \frac{\left(\sum_{n=1}^{\infty} g(x, p, n)(q-1)^n \right)^l}{l!}, \quad (5.5)$$

and, as a consequence,

$$h(x, p, q) = e^{ikx} \left[1 + \sum_{n=1}^{\infty} l(x, p, n) \right] \quad (5.6)$$

where

$$\begin{aligned} l(x, p, n) &= \frac{1}{n!} \sum_{s=n}^{\infty} \sum_{s_1=1}^{s-n+1} \sum_{s_2=1}^{s-s_1-n+2} \cdots \sum_{s_{n-1}=1}^{s-s_1-s_2-\cdots-s_{n-2}-1} \\ &\quad g(x, p, s_1) g(x, p, s_2) \cdots g(x, p, s_{n-1}) \times \\ &\quad g(x, p, s - s_1 - s_2 - \cdots - s_{n-1}) (q-1)^s. \end{aligned} \quad (5.7)$$

One can write the q-Laplace transform as

$$L(p, q) = [H(q - 1) - H(q - 2)] \times \left\{ H[\Re(p)] \int_0^\infty f(x) h(x, p, q) dx - H[-\Re(p)] \int_{-\infty}^0 f(x) h(x, p, q) dx \right\}. \quad (5.8)$$

6 The q-Laplace transform of the q-Gaussian

Our purpose is to calculate the q-Laplace transform of the q-Gaussian. As this becomes a too complex task in the general case, we content ourselves with a first-order expansion in powers of $q - 1$. Accordingly,

$$h(x, p, q) = e^{-px} [1 + g(x, p, 1)(q - 1)], \quad (6.1)$$

with

$$g(x, p, 1) = \frac{(px)^2}{2} - px \ln[f(x)]. \quad (6.2)$$

Then, up to first order we have for the q-Laplace transform

$$L(p, q) = [H(q - 1) - H(q - 2)] \times \left\{ H[\Re(p)] \int_0^\infty \left\{ 1 + \left\{ \frac{(px)^2}{2} - px \ln[f(x)] \right\} (q - 1) \right\} f(x) e^{-px} dx - H[-\Re(p)] \int_{-\infty}^0 \left\{ 1 + \left\{ \frac{(px)^2}{2} - px \ln[f(x)] \right\} (q - 1) \right\} f(x) e^{-px} dx \right\}. \quad (6.3)$$

Let $G(k)$ and $G(k, \beta)$ be given by

$$G(p) = \left\{ H[\Re(p)] \int_0^\infty f(x) e^{-px} dx - H[-\Re(p)] \int_{-\infty}^0 f(x) e^{-px} dx \right\}, \quad (6.4)$$

$$G(p, \beta) = \left\{ H[\Re(p)] \int_0^\infty [f(x)]^\beta e^{-px} dx - H[-\Re(p)] \int_{-\infty}^0 [f(x)]^\beta e^{-px} dx \right\}. \quad (6.5)$$

We write the q-Laplace transform in the form

$$L(p, q) = [H(q-1) - H(q-2)] \times \\ G(p) + \left[\frac{p^2}{2} \frac{\partial^2}{\partial p^2} G(p) + p \frac{\partial}{\partial p} \frac{\partial}{\partial \beta} G(p, \beta) \right]_{\beta=1} (q-1). \quad (6.6)$$

Let $f(x)$ be the q-Gaussian

$$f(x) = C_{q'} [1 + (q' - 1) \alpha x^2]^{\frac{1}{1-q'}}, \quad (6.7)$$

where

$$C_{q'} = \frac{\sqrt{(q' - 1) \alpha}}{B\left(\frac{1}{2}, \frac{1}{q' - 1} \frac{1}{2}\right)} \quad q' \neq 1, \quad (6.8)$$

$$C_1 = \sqrt{\frac{\alpha}{\pi}}. \quad (6.9)$$

Using results of [25] we obtain

$$G(p, q') = H[\Re(p)] C_{q'} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{2-q'}{1-q'}\right)}{[(q' - 1) \alpha]^{\frac{1}{1-q'}}} \left[\frac{2}{(q' - 1) \alpha p} \right]^{\frac{2-q'}{1-q'} - \frac{1}{2}} \times$$

$$\begin{aligned}
& \left\{ \mathbf{H}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) \right\} - \\
& H[-\Re(p)] C_{q'} \frac{\sqrt{\pi}}{2} \frac{\Gamma \left(\frac{2-q'}{1-q'} \right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}} \left[\frac{2}{(1-q')\alpha p} \right]^{\frac{2-q'}{1-q'}-\frac{1}{2}} \times \\
& \left\{ \mathbf{H}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(1-q')\alpha} \right) - \mathbf{N}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(1-q')\alpha} \right) \right\}, \quad (6.10)
\end{aligned}$$

and

$$\begin{aligned}
G(p, q', \beta) &= H[\Re(p)] C_{q'}^{\beta} \frac{\sqrt{\pi}}{2} \frac{\Gamma \left(\frac{\beta+1-q'}{1-q'} \right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}} \left[\frac{2}{(q'-1)\alpha p} \right]^{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \times \\
& \left\{ \mathbf{H}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) \right\} - \\
& H[-\Re(p)] C_{q'}^{\beta} \frac{\sqrt{\pi}}{2} \frac{\Gamma \left(\frac{\beta+1-q'}{1-q'} \right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}} \left[\frac{2}{(1-q')\alpha p} \right]^{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \times \\
& \left\{ \mathbf{H}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(1-q')\alpha} \right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(1-q')\alpha} \right) \right\}, \quad (6.11)
\end{aligned}$$

where \mathbf{H} and \mathbf{N} are the Struve and Neumann functions, respectively. The

q-Laplace transform of the q-Gaussian is now

$$\begin{aligned}
L(p, q, q') &= [H(q-1) - H(q-2)] \times \\
G(p, q') &+ \left[\frac{p^2}{2} \frac{\partial^2}{\partial p^2} G(p, q') + k \frac{\partial}{\partial p} \frac{\partial}{\partial \beta} G(p, q', \beta) \right]_{\beta=1} (q-1). \quad (6.12)
\end{aligned}$$

If instead of using the bilateral q-Laplace transform we use the unilateral

one, we should replace (6.10) and (6.11), respectively, by

$$G(p, q') = H[\Re(p)] C_{q'} \frac{\sqrt{\pi}}{2} \frac{\Gamma \left(\frac{2-q'}{1-q'} \right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}} \left[\frac{2}{(q'-1)\alpha p} \right]^{\frac{2-q'}{1-q'}-\frac{1}{2}} \times$$

$$\left\{ \mathbf{H}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{2-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) \right\}, \quad (6.13)$$

and

$$\begin{aligned} G(p, q', \beta) &= H[\Re(p)] C_{q'}^\beta \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\beta+1-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}} \left[\frac{2}{(q'-1)\alpha p} \right]^{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \times \\ &\left\{ \mathbf{H}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \left(\frac{p}{(q'-1)\alpha} \right) \right\}. \end{aligned} \quad (6.14)$$

7 The q-Laplace transform of the q-Gaussian for fixed q

In this section we deal with the q-Laplace transform of the q-Gaussian for fixed q . We have

$$\begin{aligned} L_{PR}(p, q) &= H[\Re(p)] \int_0^\infty C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \times \\ &\left\{ 1 + (1-q)ikx \left\{ C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \right\} (q-1) \right\}^{\frac{1}{1-q}} dx - \\ &H[-\Re(p)] \int_{-\infty}^0 C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \times \\ &\left\{ 1 + (q-1)px \left\{ C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \right\} (q-1) \right\}^{\frac{1}{1-q}} dx, \end{aligned} \quad (7.1)$$

$1 \leq q < 2$. Simplifying terms we obtain

$$L_{PR}(p, q) = H[\Re(p)] \int_0^\infty C_q [(q-1)\alpha x^2 + (q-1)C_q^{q-1}px + 1]^{\frac{1}{1-q}} dx -$$

$$H[-\Re(p)] \int_{-\infty}^0 C_q[(q-1)\alpha x^2 + e^{\frac{i\pi}{2}}(q-1)C_q^{q-1}kx + 1]^{\frac{1}{1-q}} dx. \quad (7.2)$$

Effecting the change of variables $\sqrt{(q-1)\alpha} x = y$, the q-Laplace transform becomes

$$L_{PR}(p, q) = \frac{H[\Re(p)]}{\sqrt{(q-1)\alpha}} \int_0^\infty C_q \left[y^2 + C_q^{q-1} \sqrt{\frac{q-1}{\alpha}} py + 1 \right]^{\frac{1}{1-q}} dy -$$

$$\frac{H[-\Re(p)]}{\sqrt{(q-1)\alpha}} \int_0^\infty C_q \left[y^2 - C_q^{q-1} \sqrt{\frac{q-1}{\alpha}} py + 1 \right]^{\frac{1}{1-q}} dy. \quad (7.3)$$

Using results given in [26] we see that (\mathbf{P}_ν^μ is the associated Legendre function)

$$\mathbf{P}_\nu^\mu(z) = \frac{2^\mu \Gamma(1-2\mu)(z^2-1)^{\frac{\mu}{2}}}{\Gamma(1-\mu)\Gamma(-\mu-\nu)\Gamma(\nu-\mu+1)} \times$$

$$\int_0^\infty (1+2tz+t^2)^{\mu-\frac{1}{2}} t^{-1-\nu-\mu} dt, \quad (7.4)$$

and we can write

$$\int_0^\infty (1+2tz+t^2)^{\mu-\frac{1}{2}} t^{-1-\nu-\mu} dt = \Gamma(-\mu) 2^{-\mu-1} (z^2-1)^{\frac{\mu}{2}} \mathbf{P}_{-\mu-1}^\mu(z), \quad (7.5)$$

Let γ, μ be given by:

$$\gamma = \frac{C_q^{q-1}}{2} \sqrt{\frac{q-1}{\alpha}} \quad \mu = \frac{1}{1-q} + \frac{1}{2}, \quad (7.6)$$

so that

$$L_{PR}(p, q) = C_q \frac{\Gamma(-\mu)}{\sqrt{(q-1)\alpha}} 2^{-\mu-1} (\gamma^2 p^2 - 1)^{\frac{\mu}{2}} \times$$

$$\{H[\Re(p)]\mathbf{P}_{-1-\mu}^{\mu}(\gamma p) - H[-\Re(p)]\mathbf{P}_{-1-\mu}^{\mu}(-\gamma p)\}, \quad (7.7)$$

which is the q-Laplace Transform of the q-Gaussian (on the complex plane) for fixed q . If instead of using the bilateral q-Laplace transform we use the unilateral one, we just obtain

$$L_{PR}(p, q) = C_q \frac{\Gamma(-\mu)}{\sqrt{(q-1)\alpha}} 2^{-\mu-1} (\gamma^2 p^2 - 1)^{\frac{\mu}{2}} H[\Re(p)] \mathbf{P}_{-1-\mu}^{\mu}(\gamma p). \quad (7.8)$$

8 The q-partition function

As it is well known, the partition function of a system is the unilateral Laplace transform of the density of states [22]. Thus, the q-partition function should be defined as the q-Laplace transform of the density of states $f(u)$, in which the complex variable is now \mathcal{B} (in place of p). We have

$$Z(\mathcal{B}, q) = [H(q-1) - H(q-2)] \times H[\Re(\mathcal{B})] \int_0^{\infty} f(u) \{1 - (1-q)\mathcal{B}u[f(u)]^{(q-1)}\}^{\frac{1}{1-q}} du. \quad (8.1)$$

If the density of states depends on q we can define it following the L_{PR} definition given in section 2.

$$Z_{PR}(\mathcal{B}, q) = [H(q-1) - H(q-2)] \times H[\Re(\mathcal{B})] \int_0^{\infty} f_q(u) \{1 - (1-q)\mathcal{B}u[f_q(u)]^{(q-1)}\}^{\frac{1}{1-q}} du. \quad (8.2)$$

For example, if the density of states is a q-exponential

$$f_q(u) = [1 + (q - 1)\alpha u]^{\frac{1}{1-q}}, \quad (8.3)$$

where $\alpha > 0$, the q-partition function is

$$Z_{PR}(\mathcal{B}, q) = [H(q - 1) - H(q - 2)] \frac{H[\Re(\mathcal{B})]}{\mathcal{B} + \alpha}. \quad (8.4)$$

An important case is $f(u) = \text{constant} = H(x)$. We face a non-degenerate energy spectrum (the one dimensional harmonic oscillator, for instance). We have then,

$$Z(\mathcal{B}, q) = [H(q - 1) - H(q - 2)] \frac{H[\Re^+(\mathcal{B})]}{(2 - q)\mathcal{B}}. \quad (8.5)$$

Conclusions

We have here developed the q-Laplace transform, thus incorporating it to the Tsallis' arsenal. We have studied its main properties and analyzed some instructive examples. The particularly important case of the q-Gaussian has been discussed in some detail. Finally, we have derived the q-partition function from the q-Laplace transform. As is also the case with the q-Fourier transform, we realize that the q-Laplace transform is essentially a transformation between equivalence classes.

9 Appendix: Tempered ultradistributions and distributions of exponential type

Many readers will surely benefit from a brief summary of the main properties of distributions of exponential type and of tempered ultradistributions.

Notations. The notations are almost textually taken from Ref. [30]. Let \mathbb{R}^n (resp. \mathbb{C}^n) be the real (resp. complex) n -dimensional space whose points are denoted by $x = (x_1, x_2, \dots, x_n)$ (resp $z = (z_1, z_2, \dots, z_n)$). We shall use the notations:

$$(a) \ x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \ ; \ \alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

$$(b) \ x \geq 0 \text{ means } x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

$$(c) \ x \cdot y = \sum_{j=1}^n x_j y_j$$

$$(d) \ |x| = \sum_{j=1}^n |x_j|$$

Let \mathbb{N}^n be the set of n -tuples of natural numbers. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, \dots, p_n)$, and p_j is a natural number, $1 \leq j \leq n$. $p + q$ stands for $(p_1 + q_1, p_2 + q_2, \dots, p_n + q_n)$ and $p \geq q$ means $p_1 \geq q_1, p_2 \geq q_2, \dots, p_n \geq q_n$. x^p entails $x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$. We shall denote by $|p| = \sum_{j=1}^n p_j$ and call D^p the differential operator $\partial^{p_1+p_2+\dots+p_n} / \partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}$.

For any natural k we define $x^k = x_1^k x_2^k \dots x_n^k$ and $\partial^k / \partial x^k = \partial^{nk} / \partial x_1^k \partial x_2^k \dots \partial x_n^k$.

The space \mathcal{H} of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any p and q , being defined [see Ref. ([30])] by means of the countable set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p, x} e^{p|x|} \left| D^q \hat{\phi}(x) \right|, \quad p = 0, 1, 2, \dots \quad (9.1)$$

The space of continuous linear functionals defined on \mathcal{H} is the space Λ_∞ of the distributions of the exponential type given by (ref.[30]).

$$T = \frac{\partial^k}{\partial x^k} [e^{k|x|} f(x)] \quad (9.2)$$

where k is an integer such that $k \geq 0$ and $f(x)$ is a bounded continuous function. In addition we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \Lambda_\infty$, where \mathcal{S} is the Schwartz space of rapidly decreasing test functions (ref[31]).

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\hat{\phi}}(x) e^{iz \cdot x} dx \quad (9.3)$$

According to ref.[30], $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call \mathcal{H} the set of all such functions.

$$\mathcal{H} = \mathcal{F} \{ \mathcal{H} \} \quad (9.4)$$

The topology in \mathcal{H} is defined by the countable set of semi-norms:

$$\|\phi\|_k = \sup_{z \in V_k} |z|^k |\phi(z)|, \quad (9.5)$$

where $V_k = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |Im z_j| \leq k, 1 \leq j \leq n\}$

The dual of \mathcal{H} is the space \mathcal{U} of tempered ultradistributions [see Ref. ([30])].

In other words, a tempered ultradistribution is a continuous linear functional defined on the space \mathcal{H} of entire functions rapidly decreasing on straight lines parallel to the real axis. Moreover, we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{U}$.

\mathcal{U} can also be characterized in the following way [see Ref. ([30])]: let \mathcal{A}_ω be the space of all functions $F(z)$ such that:

A) $F(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, \dots, |Im(z_n)| > p\}$.

B) $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geq p, |Im(z_2)| \geq p, \dots, |Im(z_n)| \geq p\}$, where $p = 0, 1, 2, \dots$ depends on $F(z)$.

Let Π be the set of all z -dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then \mathcal{U} is the quotient space

$$\mathbf{C})- \mathcal{U} = \mathcal{A}_\omega / \Pi$$

By a pseudo-polynomial we understand a function of z of the form

$$\sum_s z_j^s G(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \text{ with } G(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathcal{A}_\omega$$

Due to these properties it is possible to represent any ultradistribution as

[see Ref. ([30])]

$$F(\phi) = \langle F(z), \phi(z) \rangle = \oint_{\Gamma} F(z) \phi(z) dz \quad (9.6)$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \Gamma_n$, where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\text{Im}(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\text{Im}(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Eq. (9.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of the “Dirac formula” for ultradistributions [see Ref. ([20])]

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2) \dots (t_n - z_n)} dt \quad (9.7)$$

where the “density” $f(t)$ is such that

$$\oint_{\Gamma} F(z) \phi(z) dz = \int_{-\infty}^{\infty} f(t) \phi(t) dt. \quad (9.8)$$

While $F(z)$ is analytic on Γ , the density $f(t)$ is in general singular, so that the r.h.s. of (9.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ , $F(z)$ is bounded by a power of z [30]

$$|F(z)| \leq C|z|^p, \quad (9.9)$$

where C and p depend on F .

The representation (9.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} \{F(z) + P(z)\} \phi(z) dz = \oint_{\Gamma} F(z) \phi(z) dz + \oint_{\Gamma} P(z) \phi(z) dz$$

However,

$$\oint_{\Gamma} P(z) \phi(z) dz = 0.$$

As $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing), we obtain:

$$\oint_{\Gamma} \{F(z) + P(z)\} \phi(z) dz = \oint_{\Gamma} F(z) \phi(z) dz. \quad (9.10)$$

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